

Symmetric angular momentum coupling, the quantum volume operator and the 7-spin network: a computational perspective

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Abstract. A unified vision of the symmetric coupling of angular momenta and of the quantum mechanical volume operator is illustrated. The focus is on the quantum mechanical angular momentum theory of Wigner’s 6j symbols and on the volume operator of the symmetric coupling in spin network approaches: here, crucial to our presentation are an appreciation of the role of the Racah sum rule and the simplification arising from the use of Regge symmetry. The projective geometry approach permits the introduction of a symmetric representation of a network of seven spins or angular momenta. Results of extensive computational investigations are summarized, presented and briefly discussed.

1 Introduction

This paper extends the theory presented in [1,2] and provides numerical documentation.

The attention towards the symmetric coupling of angular momenta started half a century ago [3,4]: in the late Nineties, an essentially equivalent problem attracted the keen interest in spin network theories, of great relevance in current approaches to quantum gravity, see references in [1,2], where of extreme importance is the formulation of a quantum mechanical volume operator and the study of its properties. The intimate connection between the two formulations was later established [5].

Mathematical advances regarded relationships with continuous and discrete orthogonal polynomials, their Askey scheme classification, the quadratic and cubic algebras (see [6,7]), the Leonard pairs and triples (see [8] and later papers by Terwilliger). Progress in angular momentum theory has mainly regarded semiclassical asymptotics of $3nj$ symbols [9,10,11,12,13,14,15,16,17,18], also illuminating regarding the geometrical aspects. For the latter in this paper we will particularly make reference to projective viewpoints [19,20,21,22,23,24,25]).

Applications to atomic, molecular and nuclear physics have been intensive; computational aspects have been discussed recently (see previous papers in this series [26,27,28]). At times, the investigators in these areas carried out their research without exchanges among them and using different formalisms, making it difficult the appreciation and utilization from communities of applied physical, chemical and in general computational scientists. Here we try to partially fill this gap by giving the basic formulations, and then illustrations of the perspective emerging from very recent progress through presentation and discussion of computational results. Next section 2 outlines the relevant formulation; section 3 reports results of calculations of caustics and ridges, of interest for semiclassical analysis, and of the potentials U^+ and U^- for the volume operator, which acts democratically on the 7-spin network. Concluding remarks are in section 4 .

2 Basic theory

The focus in this section is on Racah sum rule formula and an account follows, inspired by the hexagonal representation in Fig. 2i in Ref. [29], see the related Eq. (6) there and also for example the corresponding equations in the comprehensive compilation [30]. In these formulas three $6j$ symbols appear involving four angular momenta (or spins) j_a, j_b, j_c, j_d , also denoted when convenient simply a, b, c, d ; to them the previous convention [27,28] is conveniently applied and extended: first, the choice of proper Regge quadrilaterals runs as before, as that of taking the set containing the minimum of eight combinations, individuating a ; then, we can arrange the four spins according to $a \leq b \leq c \leq d$. We introduce here the notation x, y, z for the three intermediates, which are the diagonals of possible quadrilaterals.

We have that x, y, z take $2a + 1$ values, and

- x is $j_{ab} = j_{cd}$, range $[b - a, b + a]$
- y is $j_{ac} = j_{bd}$, range $[c - a, c + a]$
- z is $j_{ad} = j_{bc}$, range $[d - a, d + a]$

Specifically, x and z are the previous j_{12} and j_{23} [27,28], or ℓ and $\tilde{\ell}$ in the previous paper on the volume operator [29], and y is the *seventh* Racah hidden momentum according to Labarthe [25].

Based again on *e.g.* references [29] and [30] we list here the three defining properties of the $6j$ symbols, according to [19]:

1. *Orthonormality*:

$$\Sigma_x(2x+1) \begin{Bmatrix} a & b & x \\ c & d & y \end{Bmatrix} \begin{Bmatrix} c & d & x \\ a & b & y' \end{Bmatrix} = \frac{\delta_{yy'}}{2y'+1} \{a \ d \ y\} \{b \ c \ y\} \quad (1)$$

where $\{\dots\}$ means that the three entries obey the triangular relationship.

2. *Additivity*: (Racah sum rule)

$$\Sigma_x(-1)^{z+y+x}(2x+1) \begin{Bmatrix} a & b & x \\ c & d & y \end{Bmatrix} \begin{Bmatrix} c & d & x \\ b & a & y \end{Bmatrix} = \begin{Bmatrix} c & a & y \\ d & b & z \end{Bmatrix} \quad (2)$$

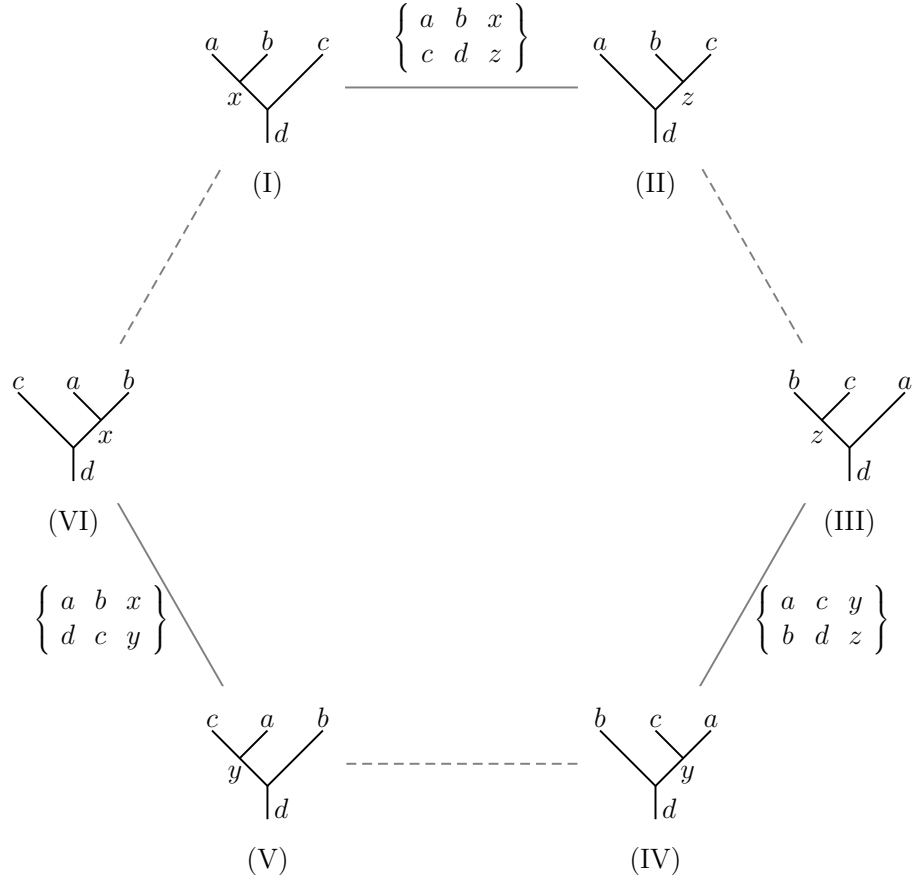


Fig. 1. The hexagonal illustration of the Racah sum rule, Eqs. 2 and 3, showing relationship among the three involved $6j$ symbols, indicating for each the corresponding changes in coupling schemes. Transformations shown as dotted lines involve simple phases only [29].

3. *Associativity*: The third (and last defining) property is the Biedenharn-Elliot identity. It provides the associative relationship schematically visualized as the pentagons in [29]. We remark here that from a projective viewpoint, it makes the underlying geometry Desarguesian [19,20,21].

Using (1), Eq. (2) becomes ([30], eq. (23) p. 467)

$$\begin{aligned} \Sigma_{xy}(-1)^{x+y+z}(2x+1)(2y+1) \begin{Bmatrix} a & b & x \\ c & d & z \end{Bmatrix} \begin{Bmatrix} a & c & y \\ d & b & x \end{Bmatrix} \begin{Bmatrix} a & d & z' \\ b & c & y \end{Bmatrix} \\ = \frac{\delta_{zz'}}{2z+1} \{a \ d \ z\} \{b \ c \ z\} \end{aligned} \quad (3)$$

The hexagonal illustration of the formulas, which involve seven angular momenta, is in Figure 1 (see also sec. 3.1 of [29]).

The volume operator [1,2]

$$\hat{K} = \mathbf{J}_a \cdot (\mathbf{J}_b \times \mathbf{J}_c) = \mathbf{J}_b \cdot (\mathbf{J}_c \times \mathbf{J}_d) = \mathbf{J}_a \cdot (\mathbf{J}_d \times \mathbf{J}_c) = \mathbf{J}_a \cdot (\mathbf{J}_b \times \mathbf{J}_d) \quad (4)$$

acts on a, b, c, d independently of the order: we can expand its eigenfunctions in one of the three alternative $|x\rangle, |y\rangle$, and $|z\rangle$ bases [3]. We have of course the same spectrum for any of the three choices, and the wavefunctions are connected by orthogonal transformations involving the three $6j$ symbols in Eqs. 2 and 3, and in Fig. 1.

3 Results, exhibited as sequences of images

3.1 Caustics and Ridges

For five sets of the a, b, c, d parameter, chosen in line with previous papers in this series [26,27,28], Figs. 2-6 illustrate caustics and ridges by means of three panels for each of the three corresponding $6j$ symbols.

For each case an additional panel shows an orthogonal three dimensional projection. Plots in panels are referred to as xz , yz , and xy views, which are sections of surfaces of constant volume, that we call Piero's "eggs" (see remark (iii) in Sec. 4). Specifically, the xz $6j$, the xy $6j$, and the yz $6j$ are

$$\begin{Bmatrix} a & b & x \\ c & d & z \end{Bmatrix}, \quad \begin{Bmatrix} a & b & x \\ d & c & y \end{Bmatrix}, \quad \begin{Bmatrix} a & c & y \\ b & d & z \end{Bmatrix} \quad (5)$$

The various sets in Figs. 2-7 illustrate a generic case and permutational or Regge symmetrical cases, with parameters taken from previous papers [27,28] to permit comparisons.

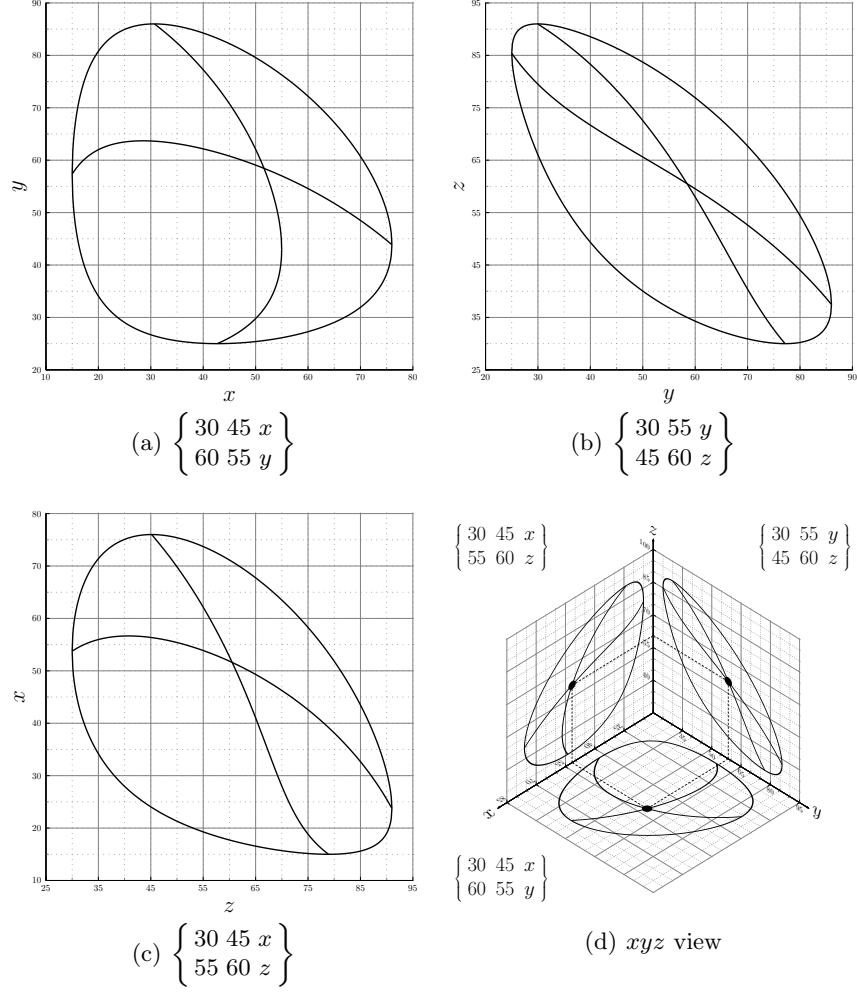


Fig. 2. Caustics and ridges of the indicated $6j$ symbols related by Racah sum rule Eqs. 2 and 3, and Fig. 1. Fig. 2(d) gives the xyz view. General case, for which entries are chosen correspondingly to Fig. 1a in Ref. [26]. Compare also with Fig. 1 in Ref. [28] where images are given of the $6j$ symbols involved in the present panel (a). See the operator volume potential functions U^+ and U^- for this case in the Fig. 7 below.

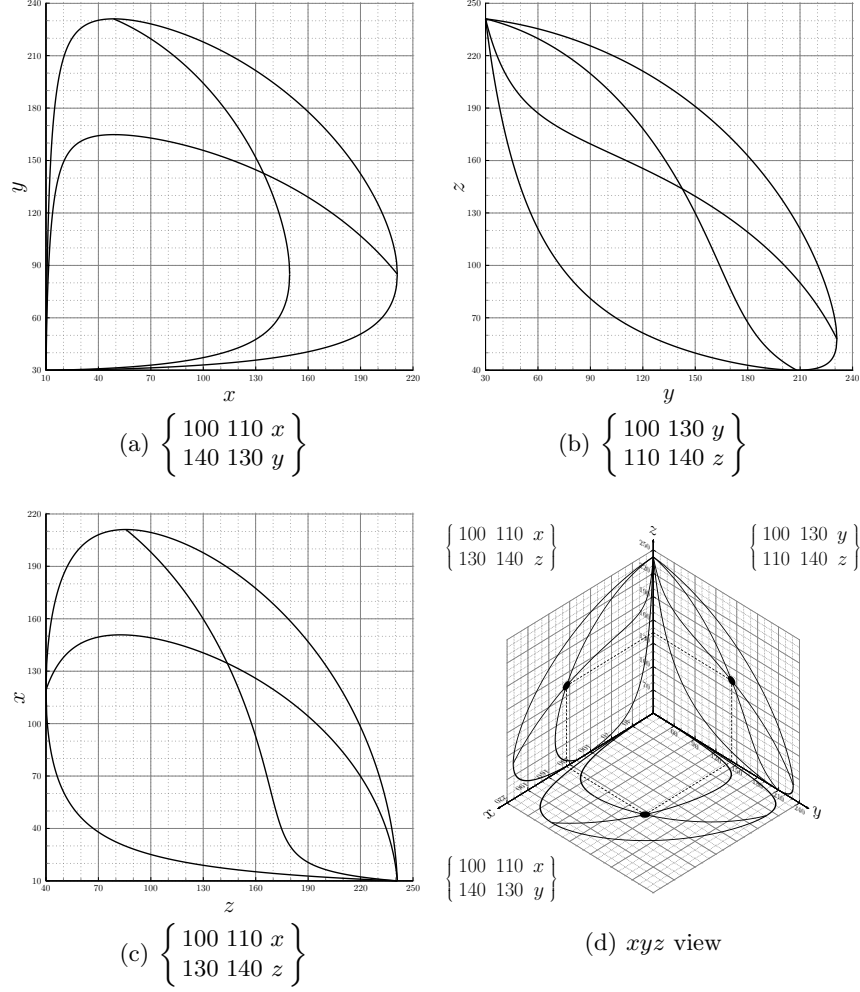


Fig. 3. As in Fig. 2, for a Regge symmetric case. Entries correspond to Figs. 1b and 1c in Ref. [26]. Compare also with Figs. 5a and 5b in Ref. [28], where images are given of the $6j$ symbols involved in the present panel (c). Note coalescences as characteristic of Regge symmetry, originating caustics cusps in the different corners of the screen for the various cases. See the operator volume potential functions U^+ and U^- for this case in the Fig. 8 below.

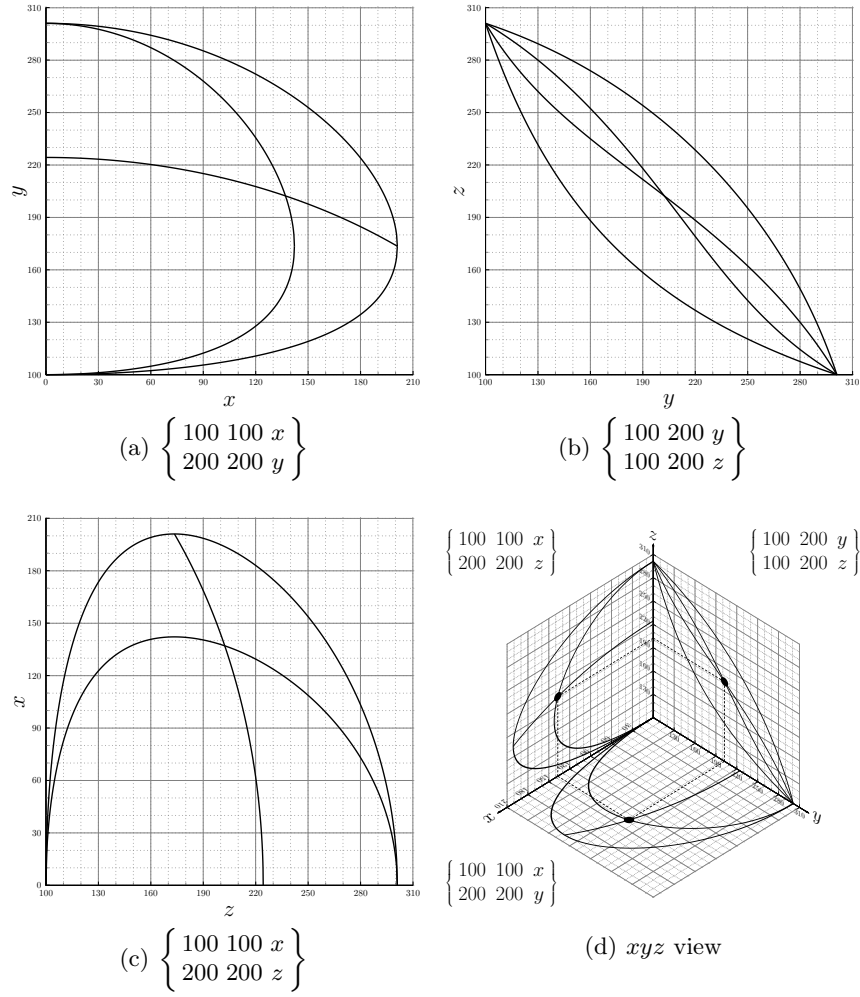


Fig. 4. As in Fig. 2, for a symmetric degenerate case. See the operator volume potential functions U^+ and U^- for this case in the Fig. 9 below.

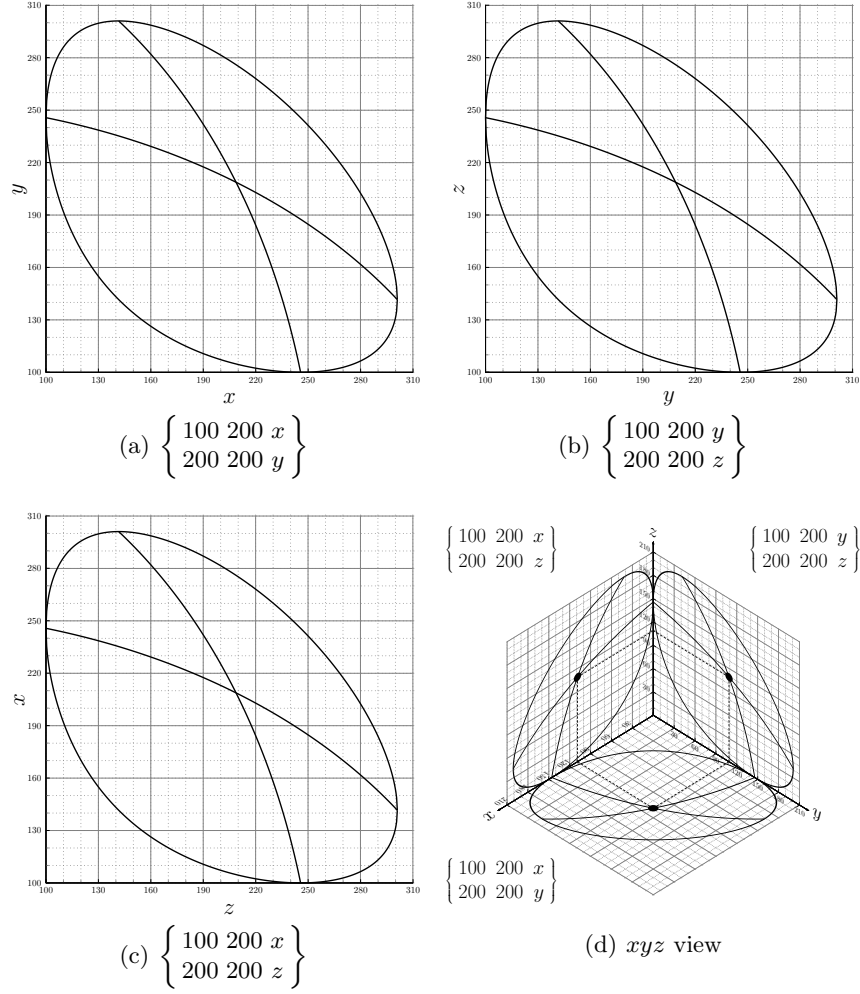


Fig. 5. As in Fig. 2, for a further symmetric case.

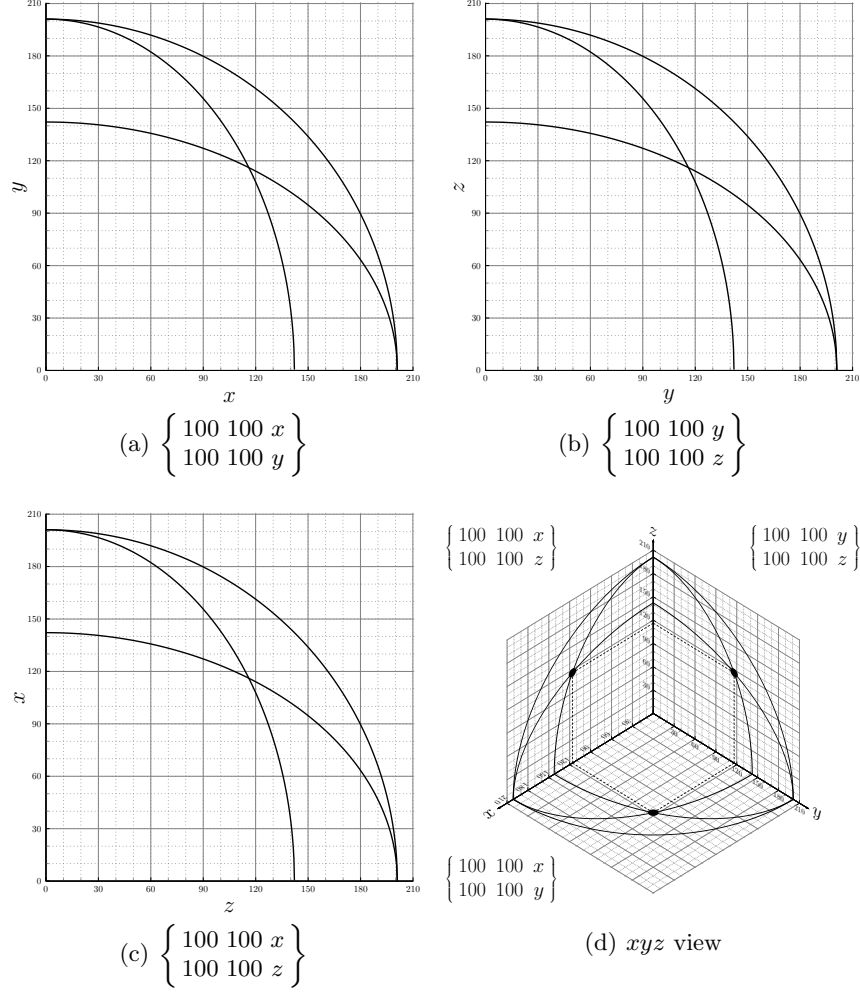


Fig. 6. As in Fig. 2, for a fully symmetric case. This is to be compared with Fig. 6 in Ref. [26] and with the images in Fig. 9 in [28]. Corresponding images for the eigenfunctions of the volume operator for the fully symmetric case are in Fig. 10 of the present paper.

3.2 Potential U^+ and U^- and eigenfunctions for the volume operator

Potential functions for the volume operator, U^+ and U^- , correspond to sections along ridges of the “egg” [1]. They are shown for some of the cases considered in Sec. 3.1 in Fig. 7-9.

Since $U^+ = -U^-$, these figures are symmetric with respect to the planes (the “screens”) of the previous figures. Finally a plot of the volume operator wavefunctions is reported in Fig. 10 for a symmetrical case.

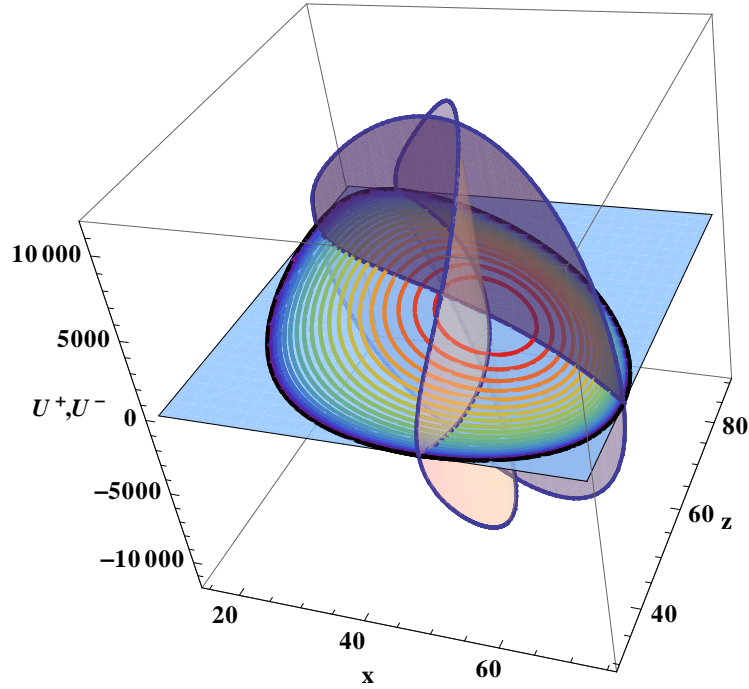


Fig. 7. Operator volume potential functions [1] U^+ for positive values and U^- for negative values (partially hidden) for $a = 30$, $b = 45$, $c = 55$, $d = 60$. This is the same as considered in Fig. 2, the horizontal zero plane here corresponding to panel (c) there.

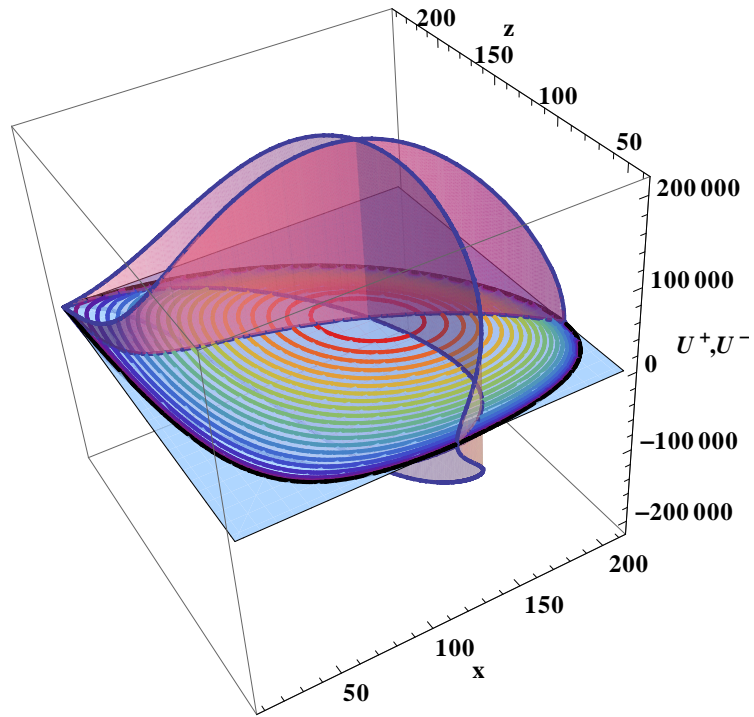


Fig. 8. Operator volume potential functions [1] U^+ for positive values and U^- (partially hidden) for negative values, for $a = 100$, $b = 110$, $c = 130$, $d = 140$. This is the case considered in Fig. 3, the horizontal zero plane here corresponding to panel (c) there.

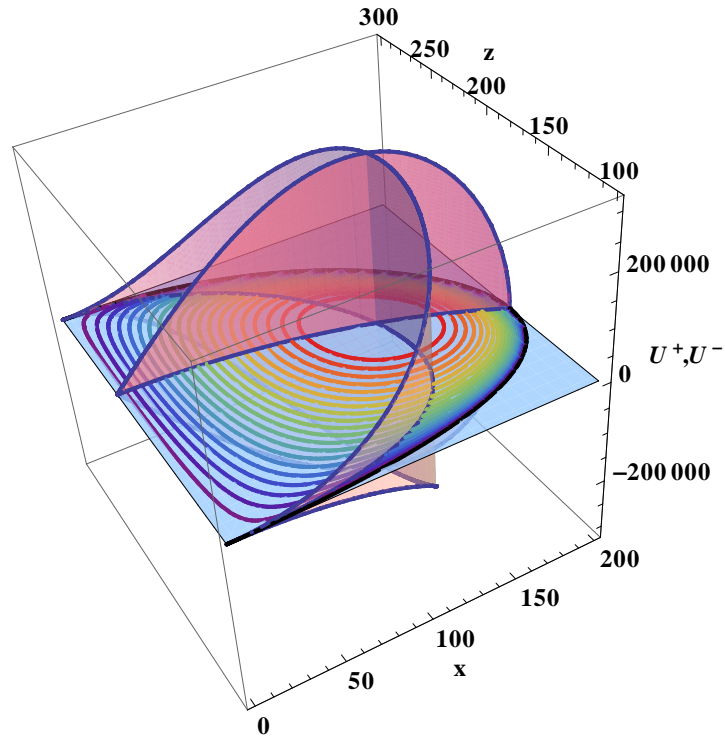


Fig. 9. As for previous figure, for $a = b = 100$, $c = d = 200$. This is the same case considered in Fig. 4, the horizontal zero plane here corresponding to panel (a) there.

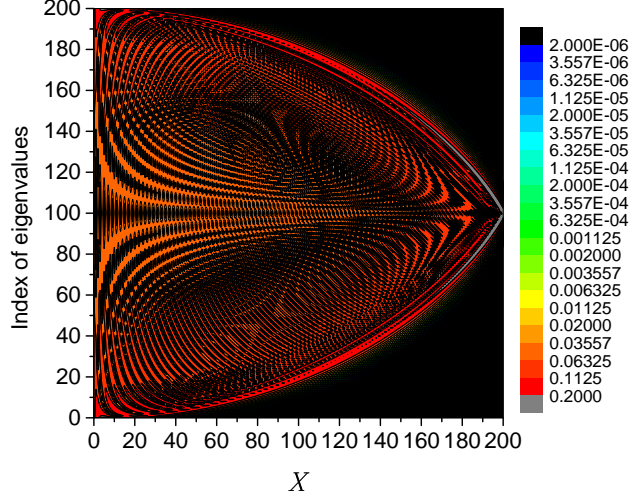


Fig. 10. The eigenfunctions of the volume operator, as defined in Ref. [1], for the fully symmetric case, for which four spins $j_1 \dots j_4$ are equal and also large. According to the nomenclature in this paper, $a = b = c = d = 100$. In the abscissa we have what we called l in Ref. [1] and x in this paper. The ordinate scale is the index of eigenvalues (0 for the lowest, etc.), that are the same in number (201) as for x . They are symmetric with respect to the 101st eigenvalue, the fast oscillations having been suppressed since here absolute values are reported. Comparison can be made with plots for these values of the parameters with the caustics and ridges of $6js$ in Fig.6 in this paper and especially with the images in Fig.9 in Ref. [28].

4 Remarks

- (i) The four angular momenta a, b, c, d , and those arising in their intermediate couplings, x, y, z , form what we define a 7-spin network, emphasizing the finite projective geometry interpretation introduced by Robinson [20], see also Refs. [22,25]. The fully symmetric nature of the network involves mapping on the Fano plane, introduced by G. Fano in 1892 as the smallest nontrivial finite projective geometry.
- (ii) From this viewpoint, the associative property in Sec. 2 permits to move from the plane to space, and to prove that the latter is Arguesian. Work is in progress to extend the theory to higher spin-networks, in connection to the morphogenetic approach of Ref. [31].
- (iii) One of the greatest mathematical achievements of the Renaissance master Piero della Francesca ($\sim 1415 - 1492$) was the discovery of the formula for the volume of a generic tetrahedron in terms of its edge lengths, used here. In one of the most beautiful of his famous masterpieces (1472), he painted an enigmatic egg at the focus of a dome designed according the newly-born

theory of perspective, of which he was one of the founding fathers. In our xy , xz , zy plots of the volume surfaces corresponding to his formula one has ovaloid, egg - like shapes: caustics, ridges and the volume operator potentials U^+ and U^- are sections of those surfaces. There is no documentation that he was aware of such a connection, which anyway motivated us to call the ovaloid as Piero's "egg".

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